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CS422

Fall 2015, Assignment #1

PROBLEM 1:
Recall the Bachmann–Landau symbols $\mathcal{O}$, Ω, Θ of asymptotic growth of functions $f,g:\mathbb{N}\to[1;\infty)$.

a) Classify the asymptotic growth of the following functions as logarithmic, polynomial, exponential, or in-between: (i) $\log(n!)$, (ii) $n^{\log\log n/\log n}$, (iii) $2^{(\log n)^2}$, (iv) $2^{\alpha(n)}$ where $\alpha(n):=\min\{m:A(m,m)\geq n\}$ for A according to Item d).

b) Describe functions f,g with neither $f\in\mathcal{O}(g)$ nor $g\in\mathcal{O}(f)$.

c) Investigate the power d of asymptotic growth $t(n)\in\Theta(n^d)$ for $t:\mathbb{N}\to\mathbb{R}$ satisfying the following recursion: $t(n)=a\cdot t(\lceil n/b\rceil)+c\cdot n$ for $1\leq a\leq b\leq c$.

d) Define $A:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ as $A(0,m):=2+m$, $A(1,m):=2m$, and recursively $A(n,m):=A\big(n-1,A(n,m-1)\big)$ where $A(n,0):=1$ for $n\geq 2$. Find explicit expressions for $A(2,m)$ and $A(3,m)$ and assert $A(m,m)\geq 2^{2^{2^m}}$ for $m\geq 4$.

PROBLEM 2:
For $n\in\mathbb{N}$ let $\ell(n)$ denote the least number of multiplications to compute the monomial x^n from x.

a) Recall why it holds $\log_2(n)\leq\ell(n)\leq 2\lfloor\log_2(n)\rfloor$.

b) Fix $\lambda\approx\log_2\log_2 n$ to be later chosen exactly. Assert that all monomials $x^0,x^1,\ldots,x^{2^\lambda-1}$ together can be calculated using a total of 2^λ multiplications.

c) Fix $a\in\mathbb{N}$ and assert that, given x^a, $x^{a\cdot 2^\lambda}$ can be calculated using another λ multiplications.

d) Adapt Horner's Method to calculate $x^{a_0+a_1\cdot 2^\lambda+a_2\cdot 2^{2\lambda}+\cdots+a_d 2^{d\cdot\lambda}}$ from $x^{a_0},\ldots,x^{a_d}$ and x^{2^λ} using $(\lambda+1)\cdot d$ multiplications.

e) Now choose $d:=\lceil\log_2 n/\lambda\rceil$ and $\lambda:\approx\log_2\log_2 n-2\log_2\log_2\log_2 n$ to improve (a).

f) Describe an algorithm asserting $\ell(2^{16}-1)\leq 19$.

PROBLEM 3:
Consider the problem of polynomial multiplication: For fixed $n\in\mathbb{N}$, on input $a_0,\ldots,a_{n-1}$ and $b_0,\ldots,b_{n-1}$, compute $c_0,\ldots,c_{2n-1}$ such that $c_0+c_1x+c_2x^2+\ldots+c_{2n-1}x^{2n-1}+x^{2n}\ =$

$$=\ (a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+x^n)\ \cdot\ (b_0+b_1x+b_2x^2+\ldots+b_{n-1}x^{n-1}+x^n)\ .$$

a) How many additions and multiplications does the high school method for this problem (aka long multiplication) incur asymptotically as $n\to\infty$?

b) Suppose w.l.o.g. (why?) that n is even. Describe a recursive/divide-and-conquer algorithm for this problem and analyze the asymptotic number of additions and multiplications it employs.

c) Verify $(A+A'\cdot X)\cdot(B+B'\cdot X)\ =\ C+C'\cdot X+C''\cdot X^2$ where $C:=A\cdot B$, $C'':=A'\cdot B'$, and $C':=(A+A')\cdot(B+B')-C-C''$. Use this to improve your algorithm from (b).