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CS422

Fall 2015, Assignment #3

PROBLEM 7:
For sets $A, B \subseteq \mathbb{N}$, their concatenation and Kleene-star are, respectively,

$$A \circ B \ := \ \big\{ \langle a, b \rangle : a \in A, b \in B \big\} \quad \text{and} \quad A^* \ := \ \big\{ \langle a_1, a_2, \ldots, a_n \rangle : n \in \mathbb{N}, a_1, \ldots, a_n \in A \big\} \ .$$

a) Prove that $\mathcal{P}$ is closed under

  i) binary union, i.e., $A, B \in \mathcal{P} \Rightarrow A \cup B \in \mathcal{P}$
  ii) intersection, i.e. $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$
  iii) complement, i.e. $A \in \mathcal{P} \Rightarrow \mathbb{N} \setminus A \in \mathcal{P}$
  iv) concatenation, i.e. $A, B \in \mathcal{P} \Rightarrow A \circ B \in \mathcal{P}$.

b) Prove that $\mathcal{PSPACE}$ is closed under
i) union, ii) intersection, iii) complement, iv) concatenation, v) Kleene-star.

c) Prove that also $\mathcal{NP}$ is closed under
i) union, ii) intersection, ~~iii) complement,~~ iv) concatenation, v) Kleene-star.

PROBLEM 8:
Prove these connections between decision problems $A \subseteq \mathbb{N}$ and function problems $f : \mathbb{N} \to \mathbb{N}$:

a) If f can be computed in polynomial time, there exists a $k \in \mathbb{N}$ with $\ell\big(f(x)\big) \leq \mathcal{O}\big(\ell(x)^k\big)$ for all $x \in \mathbb{N}$, where $\ell(x) = 1 + \lfloor \log_2 x \rfloor$ denotes the binary length.

b) If f can be computed in polynomial time, then the following decision problems lies in $\mathcal{P}$:

$$\text{Subgraph}(f) \ = \ \big\{ \langle x, y \rangle : x \in \mathbb{N}, y \leq f(x) \big\}$$

c) If Subgraph(f) is decidable in polynomial time and $\ell\big(f(x)\big) \leq \mathcal{O}\big(\ell(x)^k\big)$ holds for some k and all x, then f is computable in polynomial time.

PROBLEM 9:

a) Use the Boolean connectives $\vee, \wedge, \neg$ to construct formulae $\psi(x, y, z)$ and $\chi(x, y, z)$ such that $\psi + 2\chi$ is the binary expansion of $x + y + z$ for all $x, y, z \in \{\mathtt{0}, \mathtt{1}\}$.

b) Construct a Boolean formula $\varphi_n(\vec{x}, \vec{y}, \vec{z}, \vec{c})$ with the following property:
For $\vec{x}, \vec{y}, \vec{z} \in \{\mathtt{0}, \mathtt{1}\}^n$, $\text{bin}(\vec{z}) = z_0 + 2z_1 + \ldots + 2^{n-1} z_{n-1}$ is the binary expansion of the sum $\text{bin}(\vec{x}) + \text{bin}(\vec{y})$ iff there exists a $\vec{c} \in \{\mathtt{0}, \mathtt{1}\}^n$ such that $\varphi_n(\vec{x}, \vec{y}, \vec{z}, \vec{c}) = \mathtt{1}$.