§7 Approximation

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Input: n packets, values $v_1,\ldots v_n \in \mathbb{N}$
and weights $w_1,\ldots w_n \in \mathbb{N}$
and weight/value bounds W, V

Question: Is there a subset
$S \subseteq \{1,\ldots n\}$ s.t. values $\sum_{p\in S} v_p \geq V$
subject to weight bound $\sum_{p\in S} w_p \leq W$

Now: Find S' s.t. $\sum_{p\in S'} w_p \leq W$ and $\sum_{p\in S'} v_p \geq V\cdot(1\text{-}\varepsilon)$

Or: Find S'' s.t. $\sum_{p\in S''} w_p \leq W\cdot(1+\varepsilon)$ and $\sum_{p\in S''} v_p \geq V$

Algorithm: guaranteed approximation ratio $1\pm\varepsilon$

Discrete optimization → decision often $\mathcal{NP}$-hard
Try approximating maxim./minim. up to relative error

Dynamic Programming: *Knapsack*

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For $S\subseteq\{1,\ldots n\}$ write $w(S)=\sum_{p\in S} w_p$ and $v(S)=\sum_{p\in S} v_p$

Goal: Given W, determine $V := \max\{\, v(S) : w(S)\leq W \,\}$

Consider $T(v,m) := \min\{\, w(S) : S\subseteq\{1,\ldots m\},\ v(S)\geq v \,\}$

Note: i) $T(0,n) \leq T(1,n) \leq \ldots \leq T(V,n) \leq W < T(V+1,n)$

ii) $V = \max\{\, v : T(v,n) \leq W \,\}$

iii) $T(v,m) = 0$ for $v\leq 0$

iv) $T(v,0) = \infty$ for $v>0$

w.l.o.g. $0<w_p\leq W$

v) $T(v,m) = \min\{\, T(v,m\text{-}1)\,,\ w_m + T(v\text{-}v_m\,, m\text{-}1) \,\}$

v \ m	0	1	…	n
0	0	0	0	0
1	∞			
2	∞			
⋮	∞			T

runtime poly$(n+V)$

FPTAS for *Knapsack*

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Lemma a) For $0 \le v_p'$ it holds $V(\underline{v}) \le V(\underline{v}+\underline{v}')$

b) and for $v_p' \le \ell$: $V(\underline{v}+\underline{v}') \le V(\underline{v}) + n\cdot\ell$

c) Also, $V(k\cdot\underline{v}) = k \cdot V(\underline{v})$

$\lfloor v/k \rfloor \cdot k \le v < \lfloor v/k \rfloor \cdot k + k$

Scaling Method: Fix $k \in \mathbb{N}$ and let $v_p' := \lfloor v_p / k \rfloor$

Compute $V' := k \cdot V(v_1', \ldots, v_n')$ in time $\text{poly}(n + V/k)$. So

$V' = V(\lfloor \underline{v}/k \rfloor \cdot k) \ge V(\underline{v} - k\cdot\underline{1}) \ge V - n\cdot k = V\cdot(1 - n\cdot k/V) \overset{!}{\ge} V\cdot(1-\varepsilon)$

for $k :\approx \lfloor \varepsilon \cdot \sum_p v_p / n^2 \rfloor \le \varepsilon \cdot V/n$

$v_p \le V \Rightarrow$
$V \le \sum_p v_p \le nV$

$V/k \le O(n^2/\varepsilon + 1)$

Theorem: For every given $\varepsilon > 0$ can approximate Knapsack up to error $1-\varepsilon$ in time polynom. in $n + 1/\varepsilon$

$V(v_1, \ldots v_n) := \max\{ \sum_{p \in S} v_p : S \subseteq \{1..n\}, \sum_{p \in S} w_p \le W \}$

In-/Approximability

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Can approximate in polynomial time:

- **Knapsack** up to error 1-ε for any fixed ε>0
- **VertexCover** up to error 1+1=**2**
- **metricTSP** up to error **2**
- **Clique** up to error n, trivially

Unless $\mathcal{P}=\mathcal{NP}$, can<u>not</u> approximate

- (general) **TSP** up any to constant error
- **CLIQUE** up to error $O(n^{1-\varepsilon})$ **[Johan Håstad'96]**